

## Mechanics and Relativity for Mathematicians

- Tue Apr 1 2025: 15:00 - 17:00 -

Write your name and student number on **all** sheets. There are four problems in this exam. You can earn 90 points in total. You are allowed to bring one (hand-written) two-sided A4 page as 'cheat sheet'.

### Question 1: Planetary Motion (8+8+8)

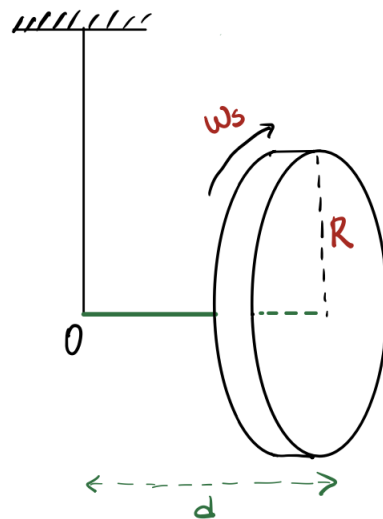
A comet is moving under Earth's gravitational influence along a parabolic trajectory. Assume that Earth is much more massive than the comet and neglect any external forces apart from gravity.

- What quantities are conserved for the comet? Under what conditions are these quantities conserved? Express these conditions in terms of the force  $\vec{F}$  and/or the potential energy  $V$ .
- What is the total energy of the comet? At which point along its trajectory does the comet attain its highest speed?
- Suppose that while the comet is approaching, Earth is suddenly replaced by another planet with a stronger gravitational field (i.e., greater mass). How would this affect the comet's trajectory? Would the new orbit still be parabolic? Explain your reasoning briefly.

### Question 2: Rigid Body Motion (4+8+4+8)

A uniform disk, with radius  $R$  and mass  $m$ , is suspended from the end of its axle, which has length  $d$ , by a rope and spun up, as shown below. It is then placed horizontally and as the disk spins, it begins to rotate around the center point  $O$ . The spin angular speed  $\omega_s$  is much greater than the precession angular speed  $\omega_p$ . Assume that the disk remains vertical and does not fall during its precession.

- What is the direction of the spin angular momentum  $\vec{L}$  of the disk at the given instant, as shown in the sketch?
- What causes precession? Find the direction of precession. Explain your answer.
- Derive the moment of inertia of a uniform disk about its center of mass.
- Find the precession angular velocity  $\omega_p$  in terms of  $m$ ,  $R$ ,  $\omega_s$ ,  $d$  and the gravitational acceleration  $g$  (Use  $I_{CM} = amR^2$  as the moment of inertia of a disk about its center of mass, where  $a$  is a constant, if you did not determine it in part c.).



**Question 3: Turntable (8+8)**

A turntable rotates counterclockwise with constant angular velocity  $\vec{\omega}$  (as viewed from above). A person stands at a radial distance  $r_0$  from the center, facing tangentially in the direction of motion. Holding the ball such that the ball, the person, and the center of the turntable are aligned, they drop it from a height. The ball is initially at rest relative to the person, and gravity  $g$  acts downward.

- (a) Determine the magnitudes and directions of the fictitious forces acting on the ball in the turntable frame at the moment the ball is released ( $t = 0$ ).
- (b) Does the ball land in front of or behind the person on the turntable as it falls? Explain your reasoning, considering the effects of the Coriolis force and centrifugal force in the rotating frame.

**Question 4: Train in a tunnel (8+8+10)**

A train and a tunnel both have proper lengths  $L$ . The train moves toward the tunnel at speed  $0.8c$ . A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, the sensor tells the bomb to disarm itself. Does the bomb explode? To answer this, follow the following steps:

- (a) In the frame of the tunnel, what is the length of the tunnel and train? What fraction of the train is inside the tunnel when the front of the train passes the far end of the tunnel?
- (b) In the frame of the train, what is the length of the tunnel and train? What fraction of the train is inside the tunnel when the far end of the tunnel passes the front of the train?
- (c) Draw a two-observer diagram on the graph paper provided, from the perspective of the tunnel frame. Include the relevant worldlines – the far and near ends of the tunnel, as well as the front and back of the train. Indicate the event when the deactivation sensor is triggered and show whether the sensor can disarm the bomb. Clearly state whether the bomb explodes in both reference frames and briefly explain your reasoning.